Lecture 6

13.3: Arc Length and Curvature

In Calc II we found the arc length of a plane curve $x(t) = f(t)$, $y(t) = g(t)$, $a \le t \le b$ as

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_a^b \sqrt{dx^2 + dy^2}$$

This was done by approximating the curve by straight lines. We can do the same thing for curves in $\mathbb{R}^3$. This leads to:

Def: The arc length of the curve $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$ is

$$\boxed{L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt}$$

There are some technical assumptions to this formula:

- $\vec{r}(t)$ does not cross itself between $t=a$ & $t=b$ (i.e., $\vec{r}$ is 1-1 on (a,b))
- $f', g',$ and h' must be continuous (i.e., $\vec{r}$ is C^1)

Notice that because

$$\|\vec{r}\,'(t)\| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

we have that the arc length can be computed as

$$\boxed{L = \int_a^b \|\vec{r}\,'(t)\|\,dt.}$$

<u>Ex</u>: Find the arc length of

$$\vec{r}(t) = \langle t, 3\cos t, 3\sin t\rangle, \quad -5 \le t \le 5$$

<u>Sol</u>: $\vec{r}\,'(t) = \langle 1, -3\sin t, 3\cos t\rangle$

$$\|\vec{r}\,'(t)\| = \sqrt{1 + 9\sin^2 t + 9\cos^2 t} = \sqrt{10}$$

So, $L = \int_{-5}^{5} \sqrt{10}\,dt = 10\sqrt{10}$

$\diamond$

A curve C need not have a unique representation by a vector function, in fact, none do. For example, $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$, $1 \le t \le 2$ is also represented by $\vec{r}_2(u) = \langle e^u, e^{2u}, e^{3t} \rangle$, $0 \le u \le \ln 2$.

$\vec{r}_1$ and $\vec{r}_2$ are called <u>parametrizations</u> of C. There is one particular parametrization we care about, and it is found as follows:

Suppose C is given by $\vec{r}(t)$, $a \le t \le b$, with $\vec{r}\,'$ continuous and $\vec{r}(t)$ traverses C exactly once. We can define the <u>arc length function</u>

$$s(t) = \int_a^t \|\vec{r}\,'(u)\| \, du$$

which tells us the distance traveled at time t. Now, suppose we can solve this equation for t in terms of s: $t = t(s)$. Then:

<u>Def</u>: The <u>arc length reparametrization</u> of $\vec{r}(t)$ is

$$\vec{r} = \vec{r}(t(s)), \quad 0 \le s \le L$$

where L = arc length of $\vec{r}$ from $t=a$ to $t=b$.

Ex: Reparametrize $\vec{r}(t) = \langle t, 3\cos t, 3\sin t \rangle$, $-5 \le t \le 5$ with respect to arc length.

Sol: First, find the arc length function:

$$s = s(t) = \int_{-5}^{t} \|\vec{r}\,'(u)\|\,du = \int_{-5}^{t} \sqrt{10}\,du$$

$$= \sqrt{10}\,t + 5\sqrt{10}$$

Solving for t gives: $t = \dfrac{s - 5\sqrt{10}}{\sqrt{10}} = t(s)$

From the last example, $L = 10\sqrt{10}$, so the bounds on s are $0 \le s \le 10\sqrt{10}$, thus the reparametrization is:

$$\vec{r}(t(s)) = \left\langle \frac{s-5\sqrt{10}}{\sqrt{10}},\ 3\cos\left(\frac{s-5\sqrt{10}}{\sqrt{10}}\right),\ 3\sin\left(\frac{s-5\sqrt{10}}{\sqrt{10}}\right) \right\rangle,$$

$$0 \le s \le 10\sqrt{10}. \quad \diamond$$

An interesting fact about arc length reparametrizations:

$$\frac{d\vec{r}}{ds} = \frac{d}{ds}\left(\vec{r}(t(s))\right) = \left(\frac{dt}{ds}\right)\vec{r}\,'(t(s)) = \frac{1}{\|\vec{r}\,'(t)\|}\,\vec{r}\,'(t(s))$$

So, $\left\|\frac{d\vec{r}}{ds}\right\| = 1$; that is, arc length reparametrizations always move with unit speed!

Curvature

Intuitively, curvature is a measure of how sharply a curve bends. Pictorally,

has larger curvature at p than

does.

Def: A parametrization $\vec{r}(t)$ is called smooth on an interval I if $\vec{r}\,'$ is continuous on I and $\vec{r}\,'(t) \neq \vec{0}$ for any $t \in I$. A curve C is called smooth if it has a smooth parametrization.

We quantify curvature as the rate of change of the unit tangent vector with respect to arc length.

In symbols, The curvature of $\vec{r}$ is:

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

Now, $\frac{d\vec{T}}{ds}$ can often be messy to compute, however, we have a trick: by the chain rule

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt} = \frac{d\vec{T}}{ds} \|\vec{r}\,'(t)\|$$

So, a more convenient formula for curvature is

$$\boxed{\kappa(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|}}$$

Ex: Find the curvature of a circle of radius a.

Sol:

Parametrize it: $\vec{r}(t) = \langle a\cos t, a\sin t\rangle$

$\vec{r}\,'(t) = \langle -a\sin t, a\cos t\rangle$, $\|\vec{r}\,'(t)\| = a$

So, $\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \langle -\sin t, \cos t\rangle$

Then, $\vec{T}\,'(t) = \langle -\cos t, -\sin t\rangle$ and $\|\vec{T}\,'(t)\| = 1$,

so $\kappa(t) = \frac{1}{a}$.

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Even that formula is more effort than needed. Another is

$$\boxed{\kappa(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}}$$

<u>Ex</u>: Find the curvature of $\vec{r}(t) = \langle \sqrt{2}\,t, e^{t}, e^{-t} \rangle$ at $(0,1,1)$.

<u>Sol</u>: First, find the relevant t value. Since $\vec{r}(0) = \langle 0, e^{0}, e^{-0} \rangle = \langle 0,1,1 \rangle$, the t value is $t=0$.

We need $\vec{r}\,'(t)$ & $\vec{r}\,''(t)$:

$\vec{r}\,'(t) = \langle \sqrt{2}, e^{t}, -e^{-t} \rangle$, $\vec{r}\,''(t) = \langle 0, e^{t}, e^{-t} \rangle$.

Now, we could either plug all of this into the formula for $\kappa(t)$ first, then plug in $t=0$, or plug in $t=0$ now, then compute $\kappa(0)$. We'll do the latter.

$\vec{r}\,'(0) = \langle \sqrt{2}, 1, -1 \rangle$, $\vec{r}\,''(0) = \langle 0,1,1 \rangle$

$$\vec{r}\,'(0) \times \vec{r}\,''(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sqrt{2} & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \langle 2, -\sqrt{2}, \sqrt{2} \rangle$$

$\|\vec{r}\,'(0)\| = \sqrt{2+1+1} = \sqrt{4} = 2$, $\|\vec{r}\,'(0) \times \vec{r}\,''(0)\| = \sqrt{4+2+2} = \sqrt{8} = 2\sqrt{2}$

So, $\kappa(0) = \dfrac{2\sqrt{2}}{2^{3}} = \dfrac{\sqrt{2}}{4}$ ◇

In the special case of a plane curve $y=f(x)$, by parametrizing it as $\vec{r}(x) = \langle x, f(x) \rangle$ we get

$$\kappa(x) = \frac{|f''(x)|}{\left(1+[f'(x)]^{2}\right)^{3/2}}$$

Frenet-Serret Frame "T-N-B Frame"

This consists of 3 vectors derived from a parametrizatio[n] $\vec{r}(t)$: $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$. We already know one of them, the other two are:

Unit Normal Vector: $\vec{N}(t) = \dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$

(requires $\|\vec{T}'(t)\| \neq 0$, equivalently $\kappa(t) \neq 0$)

Binormal Vector: $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

Since $\|\vec{T}(t)\| = 1$, we have $\vec{T}(t) \cdot \vec{T}'(t) = 0$, so $\vec{T} \perp \vec{N}$. By definition of $\times$, $\vec{B} \perp \vec{T}, \vec{N}$, so the three vectors are all orthogonal to each other. Thus since $\|\vec{T}\| = \|\vec{N}\| = 1$, we have $\|\vec{B}\| = \|\vec{T} \times \vec{N}\| = \|\vec{T}\|\,\|\vec{N}\| \sin\frac{\pi}{2} = 1$, thus all of $\vec{T}$, $\vec{N}$, and $\vec{B}$ are unit vectors. Sometimes, this way, $\vec{N}$ & $\vec{B}$ are hard to compute, so here's an alternative way:

$$\boxed{\vec{B}(t) = \frac{\vec{r}'(t) \times \vec{r}''(t)}{\|\vec{r}'(t) \times \vec{r}''(t)\|}, \quad \vec{N}(t) = \vec{B}(t) \times \vec{T}(t).}$$

$\vec{N}$ always points in the direction the curve is bending.

$\vec{B}$ points orthogonal to the motion of the curve.

We can create some planes using $\vec{T}$, $\vec{N}$, and $\vec{B}$:

Normal Plane: This plane is perpendicular to $\vec{r}(t)$. It is determined by $\vec{N}$ & $\vec{B}$, and so has $\vec{T}$ as a vector orthogonal to it.

Osculating Plane: This plane best captures the motion of the curve. It is determined by $\vec{T}$ & $\vec{N}$, and so has $\vec{B}$ as a vector perpendicular to it.

Rectifying Plane: The plane determined by $\vec{T}$ & $\vec{B}$. We won't bother with this one.

Ex: Find $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$ for $\vec{r}(t) = \langle t, 3\cos t, 3\sin t\rangle$ and find equations for the normal and osculating planes at $(\frac{\pi}{2}, 0, 3)$.

Sol: Let's begin with computing $\vec{T}(t)$:

$\vec{r}\,'(t) = \langle 1, -3\sin t, 3\cos t\rangle$, $\|\vec{r}\,'(t)\| = \sqrt{1+9\sin^2 t + 9\cos^2 t} = \sqrt{10}$

So, $\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{1}{\sqrt{10}}\langle 1, -3\sin t, 3\cos t\rangle$.

This is pretty tame, so we can differentiate it:

$$\vec{T}\,'(t) = \frac{1}{\sqrt{10}}\langle 0, -3\cos t, -3\sin t\rangle$$

$$\|\vec{T}'(t)\| = \sqrt{0^2 + \frac{9\cos^2 t}{10} + \frac{9\sin^2 t}{10}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

Thus, $\vec{N}(t) = \dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \dfrac{\frac{1}{\sqrt{10}}\langle 0, -3\cos t, -3\sin t\rangle}{\frac{3}{\sqrt{10}}} = \langle 0, -\cos t, -\sin t\rangle$

Finally,

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}}\sin t & \frac{3}{\sqrt{10}}\cos t \\ 0 & -\cos t & -\sin t \end{vmatrix}$$

$$= \left\langle \frac{3}{\sqrt{10}}\sin^2 t + \frac{3}{\sqrt{10}}\cos^2 t, \frac{1}{\sqrt{10}}\sin t, -\frac{1}{\sqrt{10}}\cos t \right\rangle$$

$$= \frac{1}{\sqrt{10}}\langle 3, \sin t, -\cos t\rangle$$

The t-value corresponding to $(\frac{\pi}{2}, 0, 3)$ is $t = \frac{\pi}{2}$.

So, we have: $\vec{T}(\frac{\pi}{2}) = \left\langle \frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}, 0 \right\rangle$ and $\vec{B}(\frac{\pi}{2}) = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0 \right\rangle$

Normal Plane: Use $\vec{T}(\frac{\pi}{2})$:

$$\left\langle \frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}, 0 \right\rangle \cdot \left\langle x - \frac{\pi}{2}, y - 0, z - 3 \right\rangle = 0$$

Osculating Plane: Use $\vec{B}(\frac{\pi}{2})$:

$$\left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0 \right\rangle \cdot \left\langle x - \frac{\pi}{2}, y - 0, z - 3 \right\rangle = 0$$

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A comment on the normal & osculating planes:

Recall that we only need a vector which is perpendicular to the plane to find an equation for it, in particular, the length of the vector doesn't matter. So, easier vectors to find to use are:

Normal Plane: use $\vec{r}\,'(t)$

Osculating Plane: use $\vec{r}\,'(t) \times \vec{r}\,''(t)$

Lecture 7

13.4 - Motion in Space

Suppose a particle moves along a trajectory $\vec{r}(t)$.

Its velocity is $\vec{v}(t) = \vec{r}\,'(t)$,

acceleration is $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$,

and speed is $\|\vec{v}(t)\|$, which I will denote by ν.

Ex: A particle has acceleration function

$$\vec{a}(t) = 4t\,\hat{\imath} + 6\sin t\,\hat{\jmath} + e^t\,\hat{k}.$$

If its initial velocity is $\vec{v}(0) = 3\hat{\jmath}$ and its initial position is $\vec{r}(0) = \vec{0}$, find its position function.

Sol: $\vec{v}(t) = \int \vec{a}(t)\,dt = (2t^2 + C_1)\hat{\imath} + (-6\cos t + C_2)\hat{\jmath} + (e^t + C_3)\hat{k}$

$\vec{v}(0) = C_1\hat{\imath} + (-6 + C_2)\hat{\jmath} + (1 + C_3)\hat{k} = 3\hat{\jmath} \Rightarrow C_1 = 0,\ C_2 = 9,\ C_3 = -1$

$\Rightarrow \vec{v}(t) = 2t^2\hat{\imath} + (9 - 6\cos t)\hat{\jmath} + (e^t - 1)\hat{k}$